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ANALYSIS OF FRACTIONAL ORDER SIS EPIDEMIC MODEL WITH CONSTANT RECRUITMENT RATE AND VARIABLE POPULATION SIZE

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ABSTRACT

In this paper the fractional order SIS epidemic model with constant recruitment rate and variable population size has been studied. Stability analysis of equilibrium points and numerical solution of this model have been shown. It has been shown that fractional order systems can help us to reduce the errors arising from the neglected parameters in modeling real life phenomena. Finally numerical simulations have been used to check the validity of the model.

Key words: Epidemic model, Recruitment rate, Stability analysis, Population size, Numerical simulation.

1. INTRODUCTION

Epidemic models incorporate constant recruitment rate, mass action incidence rate and disease-induced death rate. Since some infections are not confirming any long lasting immunity thus they do not have any recovered state and hence individuals become susceptible again after infection. This type of epidemics can be modeled as SIS epidemic model. Here the total population N is divided into two classes such as susceptible class S and infected class I [1, 2]. Thus $N = S + I$ .The use of fractional order differential and integral operators in mathematical models has become increasingly wide-spread in recent years [3]. Recently a large amount of literatures have been developed concerning the application of fractional differential equations in nonlinear dynamics [3]. Here the objective is to establish the fact that the fractional order equations are more suitable and error free than the integer order ones in modeling biological systems where memory effects are important.

1.1 Definition The fractional integral of order $\beta \in R^+$ of the function $f(t); t > 0$ is defined as

$$I^{\beta} f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)ds ------(1)$$

And the fractional derivative of order $\alpha \in (n-1, n]$ of $f(t); t > 0$ is defined as

$$D^{\alpha} f(t) = I^{n-\alpha} D^n f(t) \; ; \; D \equiv \frac{d}{dt} ------(2)$$

2.0 Equilibrium points and their stability analysis

Let $\alpha \in (0,1]$ and consider the system [4-9]

$$D^{\alpha} y_1(t) = f_1(y_1, y_2), D^{\alpha} y_2(t) = f_2(y_1, y_2) --(3)$$

With the initial values

$y_1(0) = y_{01}$ and $y_2(0) = y_{02}$ $-----(4)$.

To evaluate the equilibrium points let $D^{\alpha} y_i(t) = 0 \Rightarrow f_i\left(y_1^{eq}, y_2^{eq}\right) = 0 \; ; i = 1,2$

From which equilibrium points y_1^{eq}, y_2^{eq} can be obtained.

To evaluate the asymptotic stability, let $y_i(t) = y_i^{eq} + \varepsilon_i(t)$.

So the equilibrium point $\left(y_1^{eq}, y_2^{eq}\right)$ is locally asymptotically stable if both the eigen values

of the Jacobian matrix $A = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix}$

Evaluated at the equilibrium point [5, 6, 8-10] satisfies

$$|\arg(\lambda_1)| > \frac{\alpha\pi}{2}, |\arg(\lambda_2)| > \frac{\alpha\pi}{2}$$

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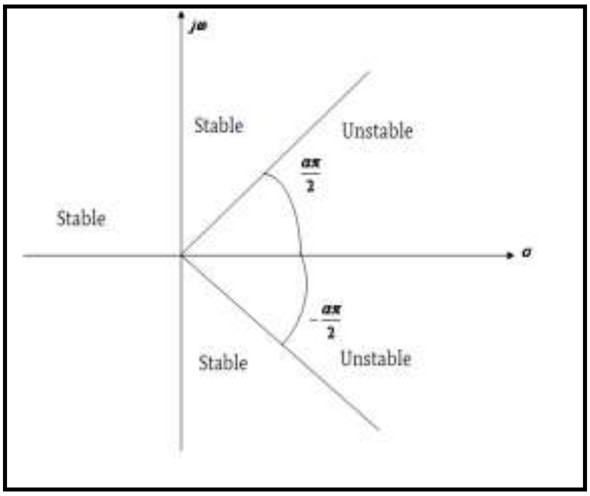

Figure 1: Stability region of the fractional-order system

Stability region of the fractional- order system with order α is illustrated in Fig. 1, in which σ and ω refer to the real and imaginary parts of the eigenvalues respectively. From Fig. 1 it is clear that the stability region of the fractional order case is greater than the stability region of the integer order case.

3.0 Fractional order SIS model

Let S (t) is the number of individuals in the susceptible class at time t, I (t) is the number of individuals who are infectious at time t. The fractional order SIS model is given by

$$\left.\begin{aligned} D^{\alpha_1} S(t) &= \Lambda - \beta SI - \mu S + \phi I \\ D^{\alpha_1} I(t) &= \beta SI - (\phi + \mu + \alpha) I \end{aligned}\right\} ----(5)$$

Where Λ is the recruitment rate of susceptible corresponding to births and immigration, μ is the per capita natural mortality rate, ϕ is the rate at which individuals becomes infectious and return to susceptible class, α represent the infection class where $0 < \alpha_1 \leq 1$. Thus together all these parameters with $N = S + I$ implies $D^{\alpha_1} N = \Lambda - \mu N - \alpha I$. Thus the total population size N may vary in time. To evaluate the equilibrium points let $D^{\alpha_1} S = 0, D^{\alpha_1} I = 0$. Then $(S_{eq}, I_{eq}) = \left(\frac{\Lambda}{\mu}, 0\right)$; (S_*, I_*) are the equilibrium points, where $S_* = \frac{1}{\beta}(\phi + \mu + \alpha)$, $I_* = \frac{\Lambda}{\mu + \alpha} - \frac{\mu(\phi + \mu + \alpha)}{\beta(\mu + \alpha)}$.

For $(S_{eq}, I_{eq}) = \left(\frac{\Lambda}{\mu}, 0\right)$ we can find

$$A = \begin{pmatrix} -\mu & -\frac{\beta\Lambda}{\mu} + \phi \\ 0 & \frac{\beta\Lambda}{\mu} - (\phi + \mu + \alpha) \end{pmatrix}$$

And its eigen values are $\lambda_1 = -\mu < 0$, $\lambda_2 = \frac{\beta\Lambda}{\mu} - (\phi + \mu + \alpha) < 0$, if

$$\frac{\beta\Lambda}{\mu} < (\phi + \mu + \alpha)$$

Hence the equilibrium point $(S_{eq}, I_{eq}) = \left(\frac{\Lambda}{\mu}, 0\right)$ is locally asymptotically stable if $\frac{\beta\Lambda}{\mu} < (\phi + \mu + \alpha) - - - - - (6)$.

For $(S_{eq}, I_{eq}) = (S_*, I_*)$,we can find

$$A = \begin{pmatrix} -\frac{\beta\Lambda}{(\mu+\alpha)} + \frac{\mu(\phi + \mu + \alpha)}{(\mu + \alpha)} - \mu & (\mu + \alpha) \\ \frac{\beta\Lambda}{(\mu+\alpha)} - \frac{\mu(\phi + \mu + \alpha)}{(\mu + \alpha)} & 0 \end{pmatrix}$$

and its corresponding eigen values are-

$$\lambda_1 = \frac{1}{2(\mu + \alpha)}\left[\begin{matrix} -(\beta\Lambda - \mu\phi) + \\ \sqrt{(\beta\Lambda - \mu\phi)^2 - 4(\mu + \alpha)^2\{\beta\Lambda - \mu(\phi + \mu + \alpha)\}} \end{matrix}\right]$$

$$\lambda_2 = \frac{1}{2(\mu + \alpha)}\left[\begin{matrix} -(\beta\Lambda - \mu\phi) - \\ \sqrt{(\beta\Lambda - \mu\phi)^2 - 4(\mu + \alpha)^2\{\beta\Lambda - \mu(\phi + \mu + \alpha)\}} \end{matrix}\right]$$

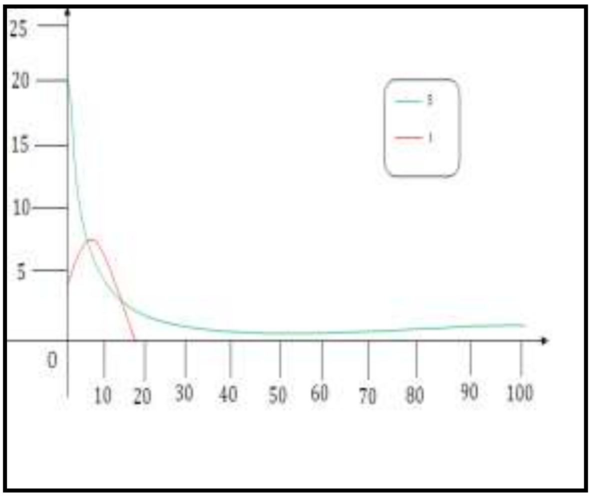

Figure 2: $\alpha_1 = 1.0$

A sufficient condition for the local asymptotic stability of the equilibrium point $(S_{eq}, Ieq) = (S_*, I_*)$ is

$$|\arg(\lambda_1)| > \frac{\alpha_1\pi}{2}, \ |\arg(\lambda_2)| > \frac{\alpha_1\pi}{2} ----(7)$$

4.0 Existence of uniformly stable solution

Let

$x_1(t) = S(t),\ x_2(t) = I(t),\ f_1(x_1(t), x_2(t))$

$= \Lambda - \beta x_1(t)x_2(t) - \mu x_1(t) + \phi x_2(t)$

and $f_2(x_1(t), x_2(t)) = \beta x_1(t)x_2(t) - (\phi + \mu + \alpha)x_2(t)$

Let $D = \{x_1, x_2 \in R : |x_i(t)| \leq a, t \in [0, T], i = 1,2\}$,

then on D we have

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$$\left|\frac{\partial}{\partial x_1} f_1(x_1, x_2)\right| \le k_1, \left|\frac{\partial}{\partial x_2} f_1(x_1, x_2)\right| \le k_2,$$

$$\left|\frac{\partial}{\partial x_1} f_2(x_1, x_2)\right| \le k_3 \text{ and } \left|\frac{\partial}{\partial x_2} f_2(x_1, x_2)\right| \le k_4$$

Where k_1, k_2, k_3 and k_4 are positive constants. This implies that each of the two functions f_1, f_2 satisfies the Lipschitz condition with respect to the two arguments x_1 and x_2 then each of the two functions f_1, f_2 is absolutely continuous with respect to the two arguments x_1 and x_2.

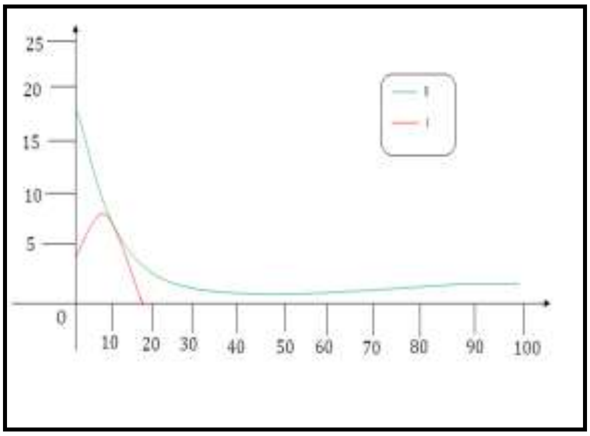

Figure 3: $\alpha_1 = 0.9$

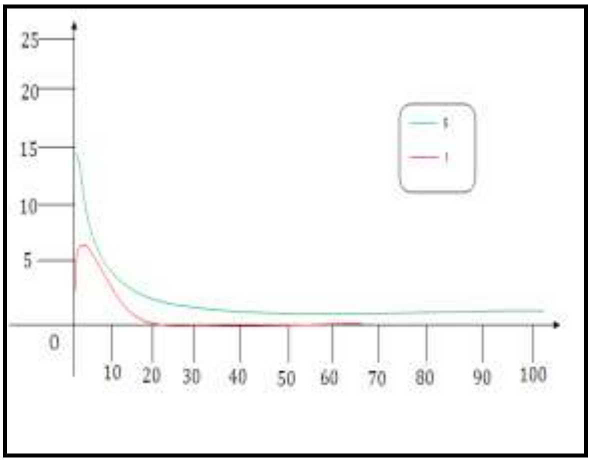

Figure 4: $\alpha_1 = 0.8$

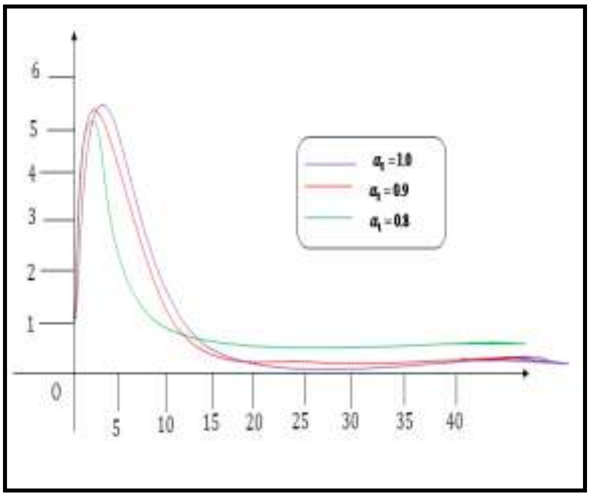

Figure 5: $\alpha_1 = 1.0, \alpha_1 = 0.9, \alpha_1 = 0.8$

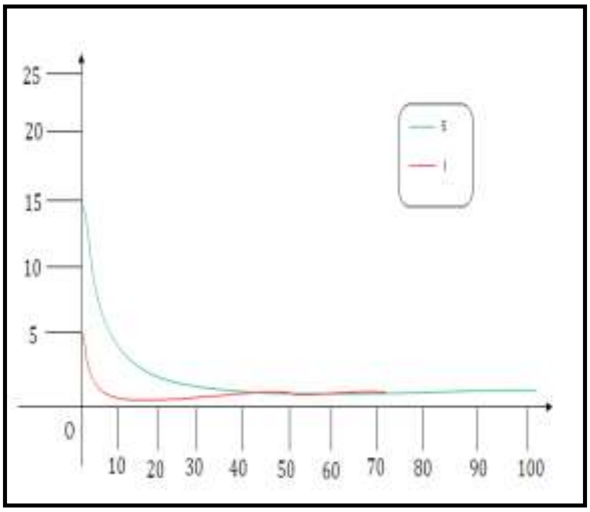

Figure 6: $\alpha_1 = 1.0$

5.0 Numerical Simulations

In this paper Adams-type predictor-corrector method has been used for the numerical solution of fractional integral equations. The key to the derivation of the method is to replace the original problem (5) by an equivalent fractional integral equation.

$$\left.\begin{aligned} S(t) &= S(0) + I^{\alpha_1}[\Lambda - \beta SI - \mu S + \phi I] \\ I(t) &= I(0) + I^{\alpha_1}[\beta SI - (\phi + \mu + \alpha)I] \end{aligned}\right\} \text{--------} (8)$$

and then apply PECE (Predict, Evaluate, Correct, Evaluate) method.

The approximate solutions are displayed in Figs. 2-6 for $S(0) = 20.0, I(0) = 1.0$ and different values of $\alpha_1 (0 < \alpha_1 \le 1)$.

In Figs. 2-5, $\Lambda = 0.1, \beta = 0.1, \mu = 0.2, \phi = 0.3$ and $\alpha = 0.1$ have been used and found that the equilibrium point $\left(\frac{\Lambda}{\mu}, 0\right) = (0.5, 0)$ is locally asymptotically stable since the condition (6) $\left(\frac{\beta\Lambda}{\mu} = 0.5 < (\phi + \mu + \alpha) = 0.6\right)$ is satisfied. In Fig. 5 it has been shown that in fractional order case the peak of the infection is reduced but the disease takes a longer time to be eradicated.

6.0 CONCLUSIONS

In this paper the stability of equilibrium points of fractional order SIS model was studied and the numerical solution of the model was also given. The main findings of this study were as follows:

(i) Fractional order systems in form of differential equations were generalizations of integer order differential equations.

(ii) Fractional order systems can help us to reduce the errors arising from the neglected parameters in modeling real life phenomena. Thus the fractional order equations are more suitable than integer order in case of mathematical modeling of epidemiological diseases.

(iii) Stability of equilibrium points was studied and numerical simulations have been used to verify the theoretical analysis.

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